T1. Particle in infinite narrow well

Let us consider a toy quantum mechanical problem: a particle with mass m is in a potential of the following form:

$$U(x) = \begin{cases} 0, & |x| > a, \\ -U_0, & |x| < a \end{cases}.$$

Now let us imagine that we reduce the width of the potential a to zero, leaving the product $U_0 a$ constant and equal to $\hbar^2 \varkappa_0 / m$.

In such a system, it turns out that there is only one bound stationary state with the energy:

$$E = -\frac{\hbar^2 \varkappa_0^2}{m}$$

A1	Find the units of $\varkappa_0$.

A2	Using the one-dimensional Schrödinger equation $$-\frac{\hbar^2}{2m}\Psi'' + U(x)\Psi = E\Psi,$$ obtain solutions for the wave function Ψ_- at $x < 0$ and Ψ_+ at $x > 0$ for the bound state. Remember that • The probability density $\rho(x)$ of finding a particle at point x is equal to $\Psi^*(x)\Psi(x)$ and therefore $$1 = \int\limits_{-\infty}^{0} \Psi_-^* \Psi_- \, \mathrm{d}x + \int\limits_{0}^{+\infty} \Psi_+^* \Psi_+ \, \mathrm{d}x$$ • The wave function is continuous $\Psi_-(0) = \Psi_+(0)$ • The wave function is defined up to phase factor $e^{i\varphi}$, where φ is a real number Sketch the graph showing the dependence of $\lvert\Psi\rvert$ on x.

A3	Calculate the probability $p(b)$ that the particle will be found at the distance less than b from the center of the well. Verify that $p(0) = 0$ and $p(\infty) = 1$.

T2. Two-level system

A simple model that can be used to describe a broad class of quantum systems is the two-level system (TLS). This is a system in which there are only two eigenstates: ground (g) and excited (e).

Let these states correspond to wave functions Ψ_e and Ψ_g. Then any state of the TLS Ψ is a superposition of Ψ_e and Ψ_g and is therefore completely characterised by two numbers α and β:

$$\Psi = \alpha\Psi_e + \beta\Psi_g,$$

where $\alpha\alpha^* + \beta\beta^* = 1$. To study the behaviour of a TLS, it is not necessary to specify its nature or even to explicitly use wave functions, so we introduce the state vector $|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$.

A1	What could the state vector be for the ground state? Remember that the wave function is definedup to the phase factor $e^{i\varphi}$, where φ is a real number.

Any operator $\hat{A}$ acting on the wave functions of the TLS is a 2×2 matrix that acts on the state vector:

$$\hat{A}|\psi\rangle = \begin{pmatrix} A_{ee} & A_{ge} \\ A_{eg} & A_{gg} \end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix}.$$

A2	Write down the Hamiltonian $\hat{\mathcal{H}}$ as a 2×2 matrix, if the energies of the ground state and excited state are equal to E_g and E_e.

A3	Let us assume that at the initial moment of time $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix}$. Using Schrödinger's equation $$i\hbar\frac{\mathrm{d}|\psi(t)\rangle}{dt} = \hat{\mathcal{H}}|\psi(t)\rangle,$$ express $|\psi(t)\rangle$ in terms of E_g and E_e.

Every physical quantity in quantum mechanics corresponds to an operator. For example, we can measure the coordinate x of a quantum system. Let the coordinate operator have the form

$$\hat{x} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

When measuring the coordinate x of the same state in quantum mechanics, we will obtain different values. The state we are interacting with can be judged only by the statistics of the measured values. For example, the average observed value of the coordinate $\langle x\rangle$ for a state with wave vector $|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ is given by the expression

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

A4	Find the dependence of the average observed $\langle x(t)\rangle$ value of the coordinate of the TLS on time if its initial state is $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix}$. Express your answer in terms of a, E_g and E_e.