T14. Rayleigh–Taylor instability

The interface between two fluids with different densities can become unstable under certain conditions. In this problem, you are asked to analyze the evolution of a small perturbation at the interface.

Throughout the entire problem, the fluids are assumed to be incompressible. Atmospheric pressure is neglected. The fluids are considered inviscid. Gravitational acceleration is denoted by g.

Surface tension can be neglected in Problems A1–A4. In all other parts of the problem, surface tension must be taken into account.

Part A. Potential flow

For irrotational flow (i.e., when streamlines do not cross), there exists a scalar function $\varphi(x, y, z, t)$, called the *velocity potential*, such that

$$\vec{v}(x, y, z) = -\nabla\varphi(x, y, z, t).$$

Note that an arbitrary function of time $f(t)$ can be added to φ, since the velocity depends only on spatial derivatives:

$$\nabla(\varphi + f(t)) = \nabla\varphi.$$

Remark 1. The minus sign in the definition of the velocity potential is not strictly necessary but is convenient for this particular problem. **Remark 2.** The introduction of the function φ is analogous to introducing a potential energy Π, where the force is expressed via the gradient of the potential:

$$\vec{F} = -\nabla\Pi.$$

A1	Write the incompressibility condition for the fluid in terms of the velocity potential φ.

We consider an incompressible, inviscid fluid. Its motion is described by the Euler equation:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v} = -\frac{\nabla P}{\rho} + \vec{g}, \qquad (1)$$

where $\vec{v}$ is the fluid velocity, ρ is the fluid density, and P is the pressure. **Remark.** In all parts of the problem, assume that the flow is irrotational and that the Euler equation applies.

Consider a layer of fluid of thickness h placed in a vessel with a flat horizontal bottom. The fluid has density ρ. Let the x-axis run along the fluid interface, and the z-axis point vertically upward. The bottom of the vessel corresponds to $z = -h$, and the undisturbed free surface of the fluid is located at $z = 0$ (see figure).

Suppose that a small perturbation arises at the free surface of the fluid, such that the velocity potential takes the form:

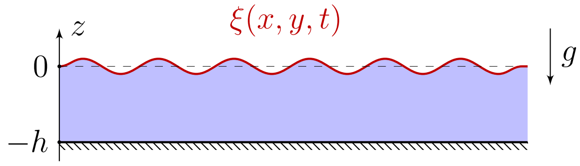
$$\varphi(x, y, z, t) = f(z)\cdot e^{i(k_x x + k_y y) - i\omega t}, \qquad (1)$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wave number of the surface wave, and ω is its frequency.

Assume that the free surface of the fluid under this perturbation is described by:

$$z = \xi(x, y, t) = \xi_0\, e^{i(k_x x + k_y y) - i\omega t},$$

where ξ_0 is the amplitude of the wave.



Remark. The assumption of small perturbations means that, to first order in the velocity v, the nonlinear term $(\vec{v} \cdot \nabla)\,\vec{v}$ in Eq. (1) can be neglected compared to $\partial \vec{v}/\partial t$. Physically, this condition can be explained as follows. Over a time interval of the order of the wave period τ, fluid particles travel a distance of the order of the wave amplitude ξ_0. Therefore, their velocity is of order $v \sim \xi_0/\tau$. The velocity v varies significantly over a time scale of order τ and over a spatial scale of order λ, where λ is the wavelength. Thus, the time derivative of the velocity is of order v/τ, and the spatial derivative is of order v/λ. Therefore, the condition:

$$(\vec{v} \cdot \nabla)\,\vec{v} \ll \frac{\partial \vec{v}}{\partial t}$$

reduces to:

$$\frac{\xi_0}{\tau} \cdot \frac{\xi_0/\tau}{\lambda} \ll \frac{\xi_0}{\tau^2}, \quad \Longrightarrow \quad \xi_0 \ll \lambda.$$

A2	Show that for *small perturbations*, to first order in the velocity v, the following equation holds: $$\frac{\partial \varphi}{\partial t} = \frac{P}{\rho} + gz + C(t),$$ where $C(t)$ is an arbitrary function of time that does not depend on spatial coordinates.

The function $C(t)$ can be set to zero, as it does not affect the velocity $\vec{v}$. The pressure can also be set to zero.

A3	What boundary condition should be imposed on $\partial\varphi/\partial z$ at the bottom of the vessel (at $z = -h$)?

A4	By applying the boundary condition for the function φ at $z = -h$ and using the incompressibility condition, find the form of the function $f(z)$. Express your answer in terms of z, h, k, and an arbitrary constant factor.

From Question A2 and the assumption of small perturbations, the following conclusion can be drawn:

$$\begin{cases} \left.\dfrac{\partial \varphi}{\partial t}\right|_{z=\xi} = g\xi, \\ \left.\dfrac{\partial \varphi}{\partial t}\right|_{z=\xi} \simeq \left.\dfrac{\partial \varphi}{\partial t}\right|_{z=0} \end{cases} \Rightarrow \quad \left.\frac{\partial \varphi}{\partial t}\right|_{z=0} \simeq g\xi.$$

A5	Express $\partial\xi/\partial t$ in terms of the derivatives of the velocity potential.

A6	By differentiating the last equation with respect to time, obtain the dispersion relation $\omega(k)$. Express your answer in terms of k, g, and h.

A7	What is the velocity c of perturbations propgation with wave vector k? Is the free surface of the fluid unstable?

Now, take into account surface tension. Let the surface tension coefficient of the fluid be σ. The Laplace pressure correction is known to be determined by the mean curvature of the surface, denoted by κ. For small perturbations ξ, the curvature κ can be approximated as:

$$P\Big|_{z=\xi} = -\sigma\kappa \approx -\sigma\Delta^{\perp}\xi = -\sigma\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right).$$

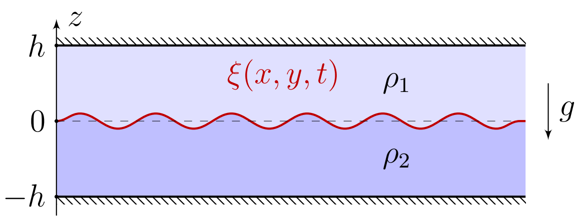
A8	Derive the dispersion relation $\omega(k)$ taking surface tension into account. Express your answer in terms of k, g, h, σ, and ρ.

A9	Is the free surface of the fluid unstable?

Part B. Interface between two fluids

Consider two immiscible fluids separated by a horizontal interface. The lower fluid has thickness h and density ρ_1, the upper fluid has thickness h and density ρ_2. Let the x-axis run along the interface between the fluids, and the z-axis point vertically upward. The flat horizontal bottom is located at $z = -h$, the flat horizontal ceiling is at $z = h$, and the interface between the two fluids corresponds to $z = 0$ (see figure).

The surface tension at the interface between the fluids is denoted by σ.



Let a small periodic perturbation $\xi(x, y, t)$ arise at the interface between the fluids, given by:

$$\xi(x, y, t) = \xi_0\, e^{i(k_x x + k_y y) - i\omega t}.$$

For each fluid, introduce its own velocity potential:

$$\varphi_1 = f_1(z)\, e^{i(k_x x + k_y y) - i\omega t}, \qquad \varphi_2 = f_2(z)\, e^{i(k_x x + k_y y) - i\omega t}.$$

B1	Using the Euler equation for small perturbations, write the expressions for the pressure $P_{1,2}$ at the interface from the perspective of each fluid. Derive an equation relating $\partial\varphi_{1,2}/\partial t$, $\rho_{1,2}$, ξ, and $\Delta^{\perp}\xi$. Follow the same reasoning as in Part A.

B2	What boundary conditions should be imposed on $\partial\varphi_{1,2}/\partial z$?

B3	Derive the dispersion relation $\omega(k)$. Express your answer in terms of σ, k, ρ_1, ρ_2, g, and h.

B4	Determine for which values of k the surface perturbation is stable. Express your answer in terms of σ, ρ_1, ρ_2, and g.