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# T7. Fresnel equations

Maxwell's equations in the dielectric material are

$$\begin{aligned} \operatorname{div}\vec{D} = 0 \quad \operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ \operatorname{div}\vec{B} = 0 \quad \operatorname{curl}\vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} \end{aligned},$$

where $\vec{D}$ is the electric displacement field related to the electric field $E$ by the $\vec{D} = \varepsilon_0 \varepsilon \vec{E}$, where $\varepsilon$ is a relative permittivity of the material. Keep in mind that $c^2 \varepsilon_0 \mu_0 = 1$, where $c$ is the speed of light.

Maxwell's equation could be solved by plane waves

$$\begin{aligned} \vec{E}(\vec{r}, t) = \mathfrak{Re}\vec{E}_0 e^{i\vec{k}\vec{r} - i\omega t} \\ \vec{B}(\vec{r}, t) = \mathfrak{Re}\vec{B}_0 e^{i\vec{k}\vec{r} - i\omega t} \end{aligned},$$

where $\mathfrak{Re}$ is a real part operator, $\vec{k}$ is a wavevector, $\omega$ is a wave frequency, $\vec{E}_0$ and $\vec{B}_0$ are amplitudes of oscillations of the electric and magnetic fields, respectively.

*Remark:* The divergence $\operatorname{div}\vec{A}$ of a vector field $\vec{A}(\vec{r})$ is a scalar value. It has the meaning of the ratio of the flux through the surface of a tiny volume to its volume and is given by

$$\operatorname{div}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

The curl $\vec{A}$ of a vector field $\vec{A}(\vec{r})$ is a vector variable and each of its components means the ratio of the circulation around some shape to its area. It's given by

$$\operatorname{curl}\vec{A} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{x}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_y & A_z \end{vmatrix} - \vec{y}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_x & A_z \end{vmatrix} + \vec{z}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_x & A_y \end{vmatrix} = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ -\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right] \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix}.$$

The Lapplacian $\Delta\vec{A}$ of the vector field $\vec{A}$ is a vector which is given by

$$\Delta\vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$$

The operators $\mathfrak{Re}$ and $\mathfrak{Im}$ return only the real and imaginary parts of a complex number:

$$\mathfrak{Re}\{a + bi\} = a, \quad \mathfrak{Im}\{a + bi\} = b$$

These operations are linear, so you can interchange it with a differentiation: $(\mathfrak{Re}\{f(x)\})' = \mathfrak{Re}\{f'(x)\}$.

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| **A1** | Substitute a plane wave into the Maxwell's equations and show that $\vec{E}_0$ and $\vec{B}_0$ are perpendicular to each other and to the $\vec{k}$. Also, write the expression for $\lvert\vec{B}_0\rvert$ in terms of $\lvert\vec{E}_0\rvert$, $\omega$ and $\lvert\vec{k}\rvert$. |

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| **A2** | With the relation $\operatorname{curl}\operatorname{curl}\vec{A} = \operatorname{grad}\operatorname{div}\vec{A} - \Delta\vec{A}$ find a wave equation for the vector $\vec{E}$: $$\Delta\vec{E} - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0.$$ What is the phase velocity $v$ of the electromagnetic wave in the medium with relative permittivity $\varepsilon$? What is the refraction index $n$ of the medium with relative permittivity $\varepsilon$? *Remark:* partial derivatives with respect to different variables could be interchanged: $$\frac{\partial^2 f(x,y)}{\partial x\partial y} = \frac{\partial^2 f(x,y)}{\partial y\partial x} \quad \Rightarrow \quad \operatorname{curl}\frac{\partial}{\partial t}\vec{A} = \frac{\partial}{\partial t}\operatorname{curl}\vec{A}$$ |

To describe the electromagnetic wave, it's usually convenient to separate the time and coordinate dependence of the electric field by introducing a vector $\tilde{E}(\vec{r})$ of the complex amplitude:

$$\vec{E}(\vec{r},t) = \mathfrak{Re}\left\{\tilde{E}(\vec{r})e^{-i\omega t}\right\}.$$

You can make sure that the Helmoltz equation holds for the complex amplitude $\tilde{E}$:

$$\Delta\tilde{E} + \frac{\omega^2}{v^2}\tilde{E} = 0.$$

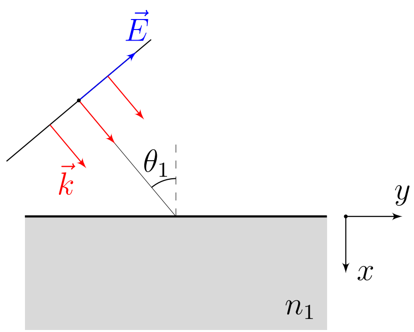
Its solution is a coordinate dependent part of the plane wave $\vec{A}e^{i\vec{k}\cdot\vec{r}}$.

Consider a plane wave with the angle of incidence $\theta_1$ at the plane interface between two dielectrics. Then the complex amplitude is

$$\tilde{E}(\vec{r}) = \begin{cases} \vec{p}_0 E_0 e^{ik_{1y}y + ik_{1x}x} + r\vec{p}_r E_0 e^{ik_{1y}y - ik_{1x}x}, & \text{for } x < 0 \\ t\vec{p}_t E_0 e^{ik_{2y}y + ik_{2x}x}, & \text{for } x > 0 \end{cases},$$

where $\vec{p}_0$, $\vec{p}_r$, $\vec{p}_t$ are unit vectors that determine the direction of the electric field oscillations, $E_0$ is the amplitude of the electric field oscillations in the incident wave, $r$ and $t$ are the amplitude coefficients of reflection and transmission, respectively.

There are two different cases of polarization: $s$- and $p$-polarized waves. The electric field $\vec{E}$ is parallel to the surface for $s$-polarization and $\vec{p}_0 = \vec{p}_r = \vec{p}_t$. The electric field $\vec{E}$ is in the plane of incidence for $p$-polarization and the vectors $\vec{p}_0$, $\vec{p}_r$, $\vec{p}_t$ are chosen so that their cross product with the wave vectors has the same direction.




The next consequence of the Maxwell's equations are boundary conditions at the plane interface between two dielectrics with relative permittivities $\varepsilon_1$ and $\varepsilon_2$. The $1, 2$ indices for each variable indicate the medium we are describing. From the Maxwell's equations with divergence it follows that the normal to the surface components are equal in pairs $\vec{D}_1$, $\vec{D}_2$ and $\vec{B}_1$, $\vec{B}_2$:

$$D_{1\perp} = D_{2\perp}, \quad B_{1\perp} = B_{2\perp}.$$

From the Maxwell's equations with curl it follows that the planar components are equal in pairs $\vec{E}_{1,2}$ and $\vec{B}_{1,2}$:

$$E_{1\|} = E_{2\|}, \quad B_{1\|} = B_{2\|}.$$

Vectors $\vec{E}$, $\vec{D}$ and $\vec{B}$ are strictly related to the complex amplitude $\tilde{E}$, so the discussed boundary conditions could be rewritten in terms of the complex amplitude. Note, that the frequency can't change during reflection or transmission because it's impossible to satisfy boundary conditions in such a case.

Let $\theta_2$ the angle of refraction.

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| **A3** | Show, that $k_{1y} = k_{2y}$, which is an equivalent of the Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$. |

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| **A4** | Using boundary conditions, show that for $s$-polarization the amplitude reflection $$r_s = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$ |

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| **A5** | Using boundary conditions, show that for $p$-polarization the amplitude reflection $$r_p = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2}$$ |

The light intensity $I$ is the average energy flux brought by the radiation. It can be found as the absolute value of the average over the period of the oscillations of the electric field of the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

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| **A6** | Show that the intensity $I$ is related to the complex amplitude $\tilde{E}$ by the following equation: $$I = \frac{c\varepsilon_0 n}{2}|\tilde{E}|^2,$$ where $n$ is the refractive index of the medium. |

The reflection coefficient $R$ is the ratio of the intensity of the reflected waves to the intensity of the incident wave.

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| **A7** | What are the values of $R_p$ and $R_s$? Express it in terms of $n_1$, $n_2$, $\theta_1$ and $\theta_2$. |

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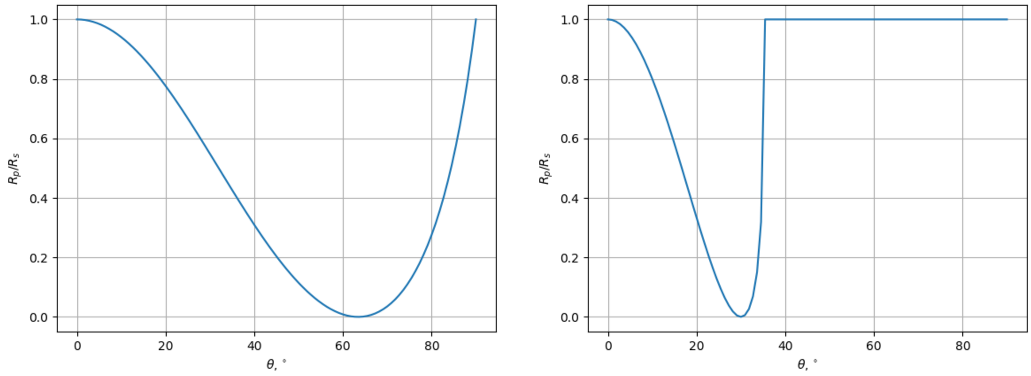
| **A8** | What is the value of the Brewster's angle $\theta_1 = \theta_\mathrm{B}$ when $R_p = 0$ i.e. the $p$-polarized light is not reflected at all. |

With fresnel.py you can plot $R_p/R_s$ as a function of incident angle $\theta_1$ for different $\varepsilon_1$ and $\varepsilon_2$.

The figure below shows the graph of $R_p/R_s$ for the data obtained in air ($n_1 = 1$) for two unknown materials $A$ and $B$ with unknown refractive indices $n_A$ and $n_B$, respectively. Note, that the refractive index of metamaterials can be almost any.




| **A9** | What are the values $n_A$ and $n_B$? |
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# T8. Cauchy's transmission equation

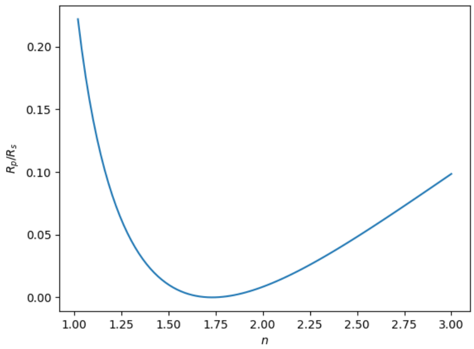
When studying the reflection properties of such a material, it's convenient to analyze the spectrum of reflection coefficients. Their dependence on the wavelength $\lambda$ is due to the fact that the relative permittivity depends on the frequency $\omega$ of the incident light according to the microstructure of the material.

Most transparent materials in the optical range have the normal dispersion: their dielectric permittivity and refractive index increase with the light frequency $\omega$. It's a common practice to approximate this dependence with empirical formulas. An example of such an empirical formula is Cauchy's transmission equation:

$$n(\lambda) = A + \frac{B}{\lambda^2},$$

where $A$ and $B$ are some constants.

If we measure the ratio $R_p/R_s$ at the only one angle of incidence, it's impossible to obtain the refractive index $n$ unambiguously, because there are two different $n$ corresponding to the value of $R_p/R_s$. The graph below shows, how the value of $R_p/R_s$ depends on $n$ (measurements are made in air) with the angle of incidence $\Phi = 60°$.




In reflectometry.py on line 10-th you can choose the value of the angle of incidence $\Phi$ and plot a graph similar to the one above. You can use the cursor (values are displayed on the bottom left) to get data from it with the cursor.

There are two series of measurements $R_p/R_s$ versus $\lambda$ for fused glass $SiO_2$ in air.

| $\Phi = 60°$ | | $\Phi = 50°$ | |
|---|---|---|---|
| $\lambda$, nm | $R_p/R_s$ | $\lambda$, nm | $R_p/R_s$ |
| 378 | 0.01311 | 378 | 0.02522 |
| 406 | 0.01352 | 406 | 0.02472 |
| 434 | 0.01384 | 434 | 0.02437 |
| 466 | 0.01415 | 466 | 0.02406 |
| 499 | 0.01442 | 499 | 0.02376 |
| 535 | 0.01464 | 535 | 0.02350 |
| 573 | 0.01485 | 573 | 0.02327 |
| 614 | 0.01500 | 614 | 0.02310 |
| 658 | 0.01515 | 658 | 0.02288 |
| 706 | 0.01537 | 706 | 0.02273 |
| 756 | 0.01553 | 756 | 0.02260 |
| 810 | 0.01568 | 810 | 0.02247 |

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| **A1** | With refractometry.py find the value of $n$ for each wavelength $\lambda$ in the table. |

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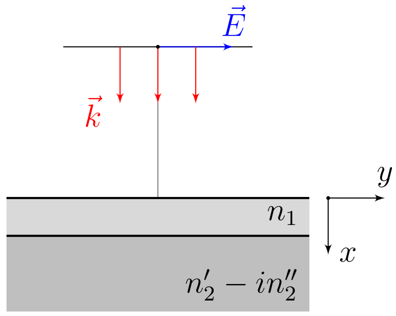
| **A2** | Plot a linear graph $n(\lambda)$ and find the values of $A$ and $B$ for fused glass. |

# T9. Sapphire deposition optical control

Let's consider the optical properties of a thin dielectric film on a substrate. The system is in air, the refractive index $n_1$ of the film is real and the refractive index $n_2 = n_2' - in_2''$ of the substrate has the imaginary part.




Let $d$ be the thickness of the film, so the complex amplitude of the electric field $\tilde{E}$ could be written as

$$\tilde{E} = E_0 \begin{cases} e^{ikx} + re^{-ikx}, & \text{for } x < 0 \\ Ae^{ikx} + Be^{-ikx}, & \text{for } 0 < x < d \\ te^{ik(x-d)}, & \text{for } x > d \end{cases}$$

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| **A1** | Using the boundary conditions for the electric field $\vec{E}$ and the magnetic field $\vec{B}$, find the amplitude reflection coefficient $r$ in such a form: $$r = \frac{a + be^{i\phi}}{ab + e^{i\phi}}.$$ |

The material of the substrate is silicon with $n_2 = 4.32 - i \cdot 0.073$ and the material of the film is sapphire $Al_2O_3$ with $n_1 \approx 1.7$ (the refractive indices are given for the wavelength $\lambda = 500$ nm).

With thin_film.py you can simulate the experiment and plot the dependence of $R = |r|^2$ as a function of the thickness $d$ of the thin film on the silicon. The refractive index $n_1$ of the film is defined in the 23rd line.

The data shown below corresponds to the cyclic chemical deposition of $Al_2O_3$ on the silicon substrate. At the beginning of the experiment, the thickness $d$ of the sapphire is zero. The intensity $I$ of the reflected light is given as a function of the number $N$ of deposition cycles.

| $N$ | $I$, a.u. | $N$ | $I$, a.u. |
|---|---|---|---|
| 0 | 390.3 | 547 | 104.5 |
| 27 | 391.0 | 574 | 88.0 |
| 55 | 385.8 | 601 | 74.5 |
| 82 | 380.5 | 629 | 64.0 |
| 109 | 374.7 | 656 | 57.4 |
| 137 | 364.9 | 683 | 54.5 |
| 164 | 358.4 | 711 | 55.3 |
| 191 | 346.8 | 738 | 60.4 |
| 219 | 336.4 | 765 | 69.2 |
| 246 | 320.0 | 792 | 81.2 |
| 273 | 304.6 | 820 | 96.7 |
| 301 | 288.2 | 847 | 113.8 |
| 328 | 268.6 | 874 | 133.4 |
| 355 | 250.6 | 902 | 153.8 |
| 383 | 228.5 | 929 | 174.0 |
| 410 | 206.7 | 956 | 197.0 |
| 437 | 186.2 | 984 | 218.0 |
| 465 | 163.5 | 1011 | 239.3 |
| 492 | 143.5 | 1038 | 259.9 |
| 519 | 122.7 | 1066 | 278.5 |

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| **A2** | What is the refractive index $n_1$ of the sapphire thin film? |

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| **A3** | Find the thickness $d_0$ of the sapphire film grown on the substrate in one deposition cycle. Consider that this value does not depend on the thickness of the already grown $Al_2O_3$. |